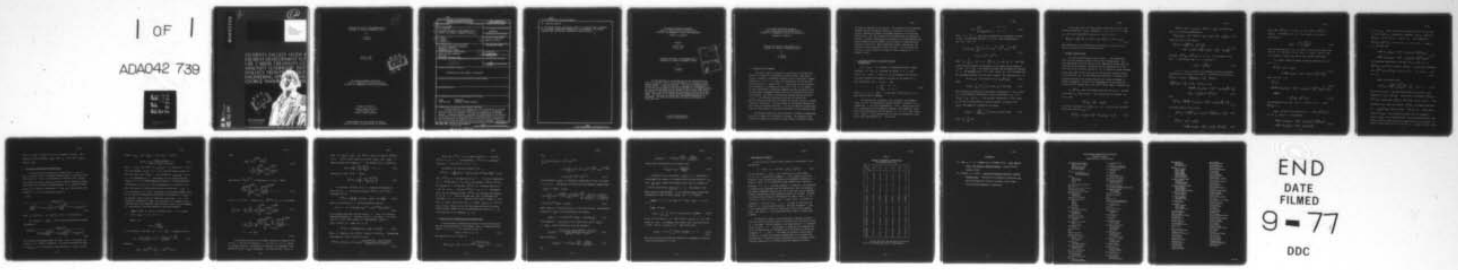
AD-A042 739 GEORGE WASHINGTON UNIV WASHINGTON D C PROGRAM IN LOG--ETC F/G 12/1
DETECTING THE SHIFT IN THE PROBABILITY OF SUCCESS IN A SERIES O--ETC(U)
JUN 77 S ZACKS, Z BARZILY N00014-75-C-0729

UNCLASSIFIED SERIAL-T-356 NL

1 OF 1

ADA042 739

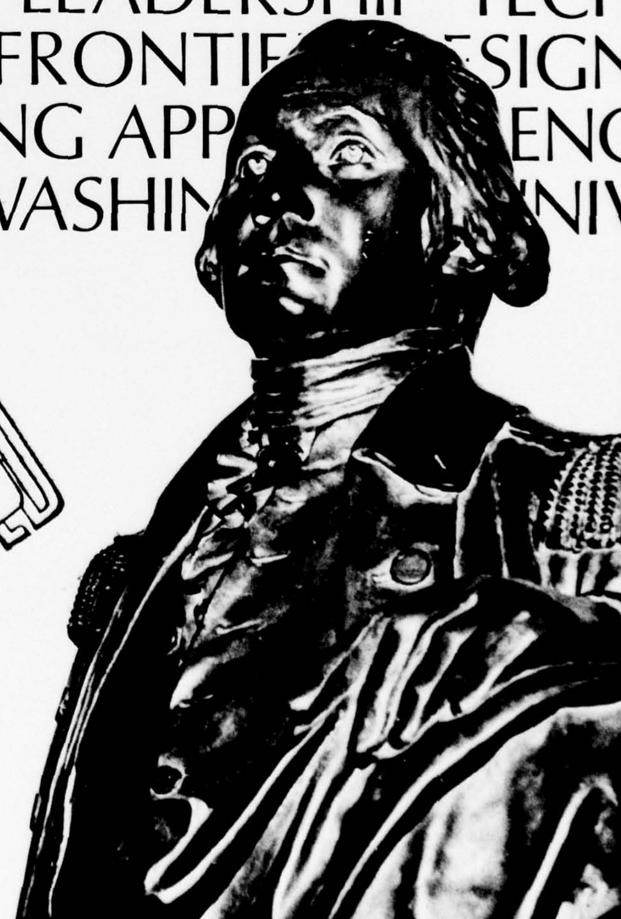


END
DATE
FILMED
9-77
DDC

(12)

THE
GEORGE
WASHINGTON
UNIVERSITY

STUDENTS FACULTY STUDY R
ESEARCH DEVELOPMENT FUT
URE CAREER CREATIVITY CO
MMUNITY LEADERSHIP TECH
NOLOGY FRONTI... ESIGN
ENGINEERING APP... ENC
GEORGE WASHIN... UNIV



INSTITUTE FOR MANAGEMENT
SCIENCE AND ENGINEERING
SCHOOL OF ENGINEERING
AND APPLIED SCIENCE

(12)

DETECTING THE SHIFT IN THE PROBABILITY OF SUCCESS IN A SERIES OF BERNOULLI TRIALS

by

S. Zacks
Z. Barzily

Serial T-356
23 June 1977

The George Washington University
School of Engineering and Applied Science
Institute for Management Science and Engineering

Program in Logistics
Contract N00014-75-C-0729
Project NR 347 020
Office of Naval Research

NONE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER (14) Serial - T-356	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) DETECTING THE SHIFT IN THE PROBABILITY OF SUCCESS IN A SERIES OF BERNOULLI TRIALS		5. TYPE OF REPORT & PERIOD COVERED SCIENTIFIC
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) S. /ZACKS, Z. /BARZILY		8. CONTRACT OR GRANT NUMBER(s) (15) N00014-75-C-0729
9. PERFORMING ORGANIZATION NAME AND ADDRESS THE GEORGE WASHINGTON UNIVERSITY, PROGRAM IN LOGISTICS, WASHINGTON, D.C. 20037		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS OFFICE OF NAVAL RESEARCH, CODE 430D, ARLINGTON, VIRGINIA 22217		12. REPORT DATE (11) 23 JUNE 1977
		13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 21p.		15. SECURITY CLASS. (of this report) NONE
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

DISTRIBUTION OF THIS REPORT IS UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

BAYES	DETECTION
STOPPING RULE	BERNOULLI RANDOM VARIABLE

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The determination of a stopping rule for the detection of the time of an increase in the success probability of a sequence of independent Bernoulli trials is discussed. Both success probabilities are assumed unknown. A Bayesian approach is applied; the distribution of the location of the shift in the success probability is assumed geometric and the success probabilities are assumed to have a known joint prior distribution. The costs involved are penalties for late or early stoppings. The nature

(Continued)

next page

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

NONE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

405 337

LB

20. Abstract (Cont'd)

of the optimal dynamic programming solution is discussed and a procedure for obtaining a suboptimal stopping rule is determined. The results indicate that the detection procedure is quite effective.

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Management Science and Engineering

Program in Logistics

Abstract
of
Serial T-356
23 June 1977

DETECTING THE SHIFT IN THE PROBABILITY OF SUCCESS IN A SERIES OF BERNOULLI TRIALS

by

S. Zacks
Z. Barzily

The determination of a stopping rule for the detection of the time of an increase in the success probability of a sequence of independent Bernoulli trials is discussed. Both success probabilities are assumed unknown. A Bayesian approach is applied; the distribution of the location of the shift in the success probability is assumed geometric and the success probabilities are assumed to have a known joint prior distribution. The costs involved are penalties for late or early stoppings. The nature of the optimal dynamic programming solution is discussed and a procedure for obtaining a suboptimal stopping rule is determined. The results indicate that the detection procedure is quite effective.

Research Sponsored by
Office of Naval Research

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Management Science and Engineering

Program in Logistics

DETECTING THE SHIFT IN THE PROBABILITY OF SUCCESS IN A SERIES OF BERNOULLI TRIALS

by

S. Zacks
Z. Barzily

1. Introduction and Summary

This paper studies the problem of controlling the success probability of a sequence of independent Bernoulli trials. More specifically, a sequence of independent Bernoulli trials starts with a success probability θ , $0 < \theta < 1$, and at an unknown epoch the success probability shifts to ϕ greater than θ . The present study is devoted to the development of a stopping rule when both θ and ϕ are unknown. We study the problem in a Bayesian framework and show the nature of the optimal dynamic programming solution when one is penalized for early or late stopping. The nature of the Bayesian optimal stopping rule when θ and ϕ are known was established previously by Sirjaev [2]. When the success probabilities are unknown the optimal solution is considerably more complicated. This paper shows how approximate solutions can be obtained and applied effectively. A series of numerical illustrations shows the effectiveness of the proposed procedure in some simulated cases.

The problem studied here was motivated by a problem of determining the epoch of change in the readiness of systems. The results of this study can be applied to a variety of other problems of applied interest.

The paper is comprised of six sections. The formulation of the Bayesian framework and the likelihood functions is carried out in Section 2. Section 3 provides the dynamic programming formulation of the optimal stopping rule. The case of known success probabilities is discussed in Section 4. Section 5 discusses the convergence of the algorithm when the success probabilities are unknown, and the results of some simulations are given in Section 6. These simulations strongly indicate that the proposed detection procedure is quite effective. It is generally very difficult to detect shifts on the basis of small sequences of $0 - 1$ Bernoulli trials.

2. Likelihood Functions, Prior and Posterior Distributions

Let $x_1, x_2, \ldots$ be a sequence of independent Bernoulli random variables, i.e., each X_i can assume the values 0 or 1, and $P[X_i=1] = \theta_i$, $i=1,2,\ldots$, where θ_i is the probability of "success." This paper considers the problem of one shift in the values of θ_i. More specifically, let $\tau = 0,1,2,\ldots$ and

$$\theta_i = \begin{cases} \theta, & \text{if } i \leq \tau \\ \phi, & \text{if } i > \tau \end{cases}, \tag{2.1}$$

where $0 < \theta < \phi < 1$. Thus τ is the epoch of shift from θ to ϕ. This epoch of shift is unknown.

The approach here is Bayesian, and accordingly, the parameter values (τ,θ,ϕ) are considered as random variables. Moreover, it is assumed that τ is priorly independent of (θ,ϕ), having a prior p.d.f. $\psi(\tau)$ concentrated on the nonnegative integers. The parameters (θ,ϕ) have a prior distribution over the simplex $0 < \theta \leq \phi < 1$, with a prior p.d.f. $h(\theta,\phi)$. In the present paper we focus our attention on a geometric prior distribution for τ of the form

$$\psi(\tau) = \begin{cases} \pi_0 & , \quad \text{if} \quad \tau = 0 \\ (1-\pi_0)p(1-p)^{\tau-1} & , \quad \text{if} \quad \tau \geq 1 \end{cases} , \tag{2.2}$$

where $0 < \pi$, $p < 1$. This prior distribution was introduced by Sirjaev in [2]. The likelihood function of (τ,θ,ϕ), given observations on $\underset{\sim}{x}_n = (x_1,\ldots,x_n)$, is

$$\begin{aligned} L(\tau,\theta,\phi;\underset{\sim}{x}_n) = & \sum_{j=0}^{n-1} I\{\tau=j\}\, \theta^{T_j} (1-\theta)^{j-T_j} \phi^{T_{n-j}^{(n)}} (1-\phi)^{n-j-T_{n-j}^{(n)}} \\ & + I\{\tau \geq n\}\, \theta^{T_n} (1-\theta)^{n-T_n} , \end{aligned} \tag{2.3}$$

where $T_j = \sum_{i=1}^{j} x_i$, $j=1,\ldots,n$, $T_{n-j}^{(n)} = T_n - T_j$ and $T_0 \equiv 0$. This likelihood function provides the information in the sample on the parameters (τ,θ,ϕ). If θ and ϕ are known we consider the likelihood as a function of τ. The <u>posterior</u> p.d.f. of (τ,θ,ϕ), given $\underset{\sim}{x}_n$, can be obtained from the likelihood function (2.3) and Bayes' formula. This posterior p.d.f. is

$$h(\tau,\theta,\phi;\underset{\sim}{x}_n) = \psi(\tau)\, h(\theta,\phi)\, L(\tau,\theta,\phi;\underset{\sim}{x}_n) \big/ D_n(\underset{\sim}{x}_n) , \tag{2.4}$$

where

$$D_n(\underset{\sim}{x}_n) = \sum_{j=0}^{\infty} \psi(j) \int_0^1 \int_\theta^1 L(j,\theta,\phi;\underset{\sim}{x}_n)\, h(\theta,\phi)\, d\phi d\theta . \tag{2.5}$$

Other functions of interest are the posterior probability of $\{\tau<n\}$ given $\underset{\sim}{x}_n$, and the posterior probability of future success given $\underset{\sim}{x}_n$, i.e., $P\{x_{n+j}=1 \mid \underset{\sim}{x}_n\}$, $j=1,2,\ldots$. The posterior probability of $\{\tau<n\}$ given $\underset{\sim}{x}_n$ is used to decide whether to stop the process. We denote it by $\pi_n(\underset{\sim}{x}_n)$ and compute it according to the formula

$$\pi_n(\underset{\sim}{x}_n) = 1 - \frac{\psi_n^*}{D_n(\underset{\sim}{x}_n)} \int_0^1 \int_\theta^1 L(n,\theta,\phi;\underset{\sim}{x}_n)\, h(\theta,\phi)\, d\phi d\theta , \tag{2.6}$$

where $\psi_n^* = \sum_{j=n}^{\infty} \psi(j)$.

We will show later that under general conditions on $h(\theta,\phi)$ the posterior probability $\pi_n(\underset{\sim}{x}_n)$ converges to 1 a.s., as $n \to \infty$. Given $\underset{\sim}{x}_n$, the future probability of success is, for all $j=1,2,\ldots$

$$P\left[x_{n+j}=1 \mid \underset{\sim}{x}_n\right] = E\left\{\theta\ I\{\tau \geq n+j\} \mid \underset{\sim}{x}_n\right\} + E\left\{\phi\ I\{\tau \leq n+j-1\} \mid \underset{\sim}{x}_n\right\} . \tag{2.7}$$

This probability can be expressed in terms similar to those of (2.6). Explicit expressions will be considered later.

3. Optimal Stopping Times

The problem of detecting the shift point τ can be approached from a decision theoretic point of view in the following terms. After each observation we have the option to stop sampling and declare that the shift has already occurred. The process is then inspected, and if the shift has not yet occurred a penalty of $\$C_2$ is imposed for undue stopping. On the other hand, if we do not stop and the shift has already occurred, we are penalized $\$C_1$ for each additional observation. We develop here a characterization of the optimal stopping rule, using the methods of dynamic programming. We start with a truncated version and impose the restriction that stopping must occur after a finite number of observations.

Let $R_n^{(j)}(\underset{\sim}{x}_n)$ denote the minimal posterior risk after n observations, given $\underset{\sim}{x}_n$, when at most j more observations are allowed. For $j=0$ we have

$$R_n^{(0)}(\underset{\sim}{x}_n) = C_2\left(1 - \pi_n(\underset{\sim}{x}_n)\right) . \tag{3.1}$$

If one more observation is allowed, i.e., $j=1$, the posterior risk associated with observing x_{n+1} is

$$\bar{R}_n^{(1)}(\underset{\sim}{x}_n) = C_1\pi_n(\underset{\sim}{x}_n) + C_2\left(1 - E\{\pi_{n+1}(\underset{\sim}{x}_{n+1}) \mid \underset{\sim}{x}_n\}\right) . \tag{3.2}$$

Moreover, since $\pi_{n+1}(\underset{\sim}{x}_{n+1}) = E\{I\{\tau \leq n\} \mid \underset{\sim}{x}_{n+1}\}$ we obtain from the law of iterated expectation that

$$E\{\pi_{n+1}(\underset{\sim}{x}_{n+1}) \mid \underset{\sim}{x}_n\} = E\{I\{\tau \leq n\} \mid \underset{\sim}{x}_n\} = \pi_n(\underset{\sim}{x}_n) + \left(1 - \pi_n(\underset{\sim}{x}_n)\right)p \ . \quad (3.3)$$

Using (3.2) and (3.3) we obtain

$$\begin{aligned} R_n^{(1)}(\underset{\sim}{x}_n) &= \min\left\{R_n^{(0)}(\underset{\sim}{x}_n) \ , \ \bar{R}_n^{(1)}(\underset{\sim}{x}_n)\right\} \\ &= C_2\left(1 - \pi_n(\underset{\sim}{x}_n)\right) + \min\left(0, \ C_1\pi_n(\underset{\sim}{x}_n) - C_2 p\left(1 - \pi_n(\underset{\sim}{x}_n)\right)\right) \ . \end{aligned} \quad (3.4)$$

Obviously, $R_n^{(1)}(\underset{\sim}{x}_n) \leq R_n^{(0)}(\underset{\sim}{x}_n)$ for all $\underset{\sim}{x}_n$. It is optimal to stop after n observations, when $j=1$, if and only if

$$\pi_n(\underset{\sim}{x}_n) \geq \frac{C_2 p}{C_1 + C_2 p} \ . \quad (3.5)$$

If $j=2$, then according to the optimality principle of dynamic programming it is optimal to stop after n observations if and only if $R_n^{(0)}(\underset{\sim}{x}_n) \leq C_1\pi_n(\underset{\sim}{x}_n) + E\left\{R_{n+1}^{(1)}(\underset{\sim}{x}_{n+1}) \mid \underset{\sim}{x}_n\right\}$. Moreover,

$$\begin{aligned} E\left\{R_{n+1}^{(1)}(\underset{\sim}{x}_{n+1}) \mid \underset{\sim}{x}_n\right\} &= C_2 \ E\{1 - \pi_{n+1}(\underset{\sim}{x}_{n+1}) \mid \underset{\sim}{x}_n\} \\ &\quad + E\left\{\min\left(0, \ C_1\pi_{n+1}(\underset{\sim}{x}_{n+1}) - pC_2\left(1 - \pi_{n+1}(\underset{\sim}{x}_{n+1})\right)\right) \mid \underset{\sim}{x}_n\right\} \\ &= C_2\left(1 - \pi_n(\underset{\sim}{x}_n)\right)(1-p) + M_n^{(1)}(\underset{\sim}{x}_n) \ , \end{aligned} \quad (3.6)$$

where

$$M_n^{(1)}(\underset{\sim}{x}_n) = E\left\{\min\left(0, \ C_1\pi_{n+1}(\underset{\sim}{x}_{n+1}) - pC_2\left(1 - \pi_{n+1}(\underset{\sim}{x}_{n+1})\right)\right) \mid \underset{\sim}{x}_n\right\} \ . \quad (3.7)$$

The minimal posterior risk is then

$$R_n^{(2)}(\underset{\sim}{x}_n) = \min\left(R_n^{(0)}(\underset{\sim}{x}_n) \ , \ C_1\pi_n(\underset{\sim}{x}_n) + E\{R_{n+1}^{(1)}(\underset{\sim}{x}_{n+1}) \mid \underset{\sim}{x}_n\}\right) \quad (3.8)$$

or

$$\begin{aligned} R_n^{(2)}(\underset{\sim}{x}_n) &= C_2\left(1 - \pi_n(\underset{\sim}{x}_n)\right) \\ &\quad + \min\left(0, \ C_1\pi_n(\underset{\sim}{x}_n) - C_2 p\left(1 - \pi_n(\underset{\sim}{x}_n)\right) + M_n^{(1)}(\underset{\sim}{x}_n)\right) \ . \end{aligned} \quad (3.9)$$

Notice that $M_n^{(1)}(\underset{\sim}{x}_n) \leq 0$ for all $\underset{\sim}{x}_n$ and therefore $R_n^{(2)}(\underset{\sim}{x}_n) \leq R_n^{(1)}(\underset{\sim}{x}_n)$. Furthermore, for $j=2$ it is optimal to stop after n observations if

$$\pi_n(\underset{\sim}{x}_n) \geq \frac{C_2p - M_n^{(1)}(\underset{\sim}{x}_n)}{C_1 + C_2p} . \tag{3.10}$$

The stopping boundary for $j=2$ [the RHS of (3.10)] is not smaller than that for $j=1$ [the RHS of (3.5)]. Thus, if $\pi_n(\underset{\sim}{x}_n) < C_2p/(C_1 + C_2p)$ it is optimal to continue and take at least one more observation.

In a similar fashion we obtain by backward induction that, for all $j \geq 2$,

$$\begin{aligned} R_n^{(j)}(\underset{\sim}{x}_n) &= C_2(1 - \pi_n(\underset{\sim}{x}_n)) \\ &\quad + \min\left(0,\ C_1\pi_n(\underset{\sim}{x}_n) - C_2p(1 - \pi_n(\underset{\sim}{x}_n)) + M_n^{(j-1)}(\underset{\sim}{x}_n)\right) , \end{aligned} \tag{3.11}$$

where, for each $i \geq 2$,

$$\begin{aligned} &M_n^{(i)}(\underset{\sim}{x}_n) \\ &= E\left\{\min\left(0,\ C_1\pi_{n+1}(\underset{\sim}{x}_{n+1}) - C_2p(1 - \pi_{n+1}(\underset{\sim}{x}_{n+1})) + M_{n+1}^{(i-1)}(\underset{\sim}{x}_{n+1})\right) \mid \underset{\sim}{x}_n\right\} . \end{aligned} \tag{3.12}$$

<u>Lemma 1</u>:

$$M_n^{(i)}(\underset{\sim}{x}_n) \leq M_n^{(i-1)}(\underset{\sim}{x}_n) \tag{3.13}$$

with probability one for all $i=1,2,\ldots$ and all $n=1,2,\ldots$, where $M_n^{(0)}(\underset{\sim}{x}_n) \equiv 0$.

<u>Proof</u>: The proof is by induction on i . Since $M_n^{(1)}(\underset{\sim}{x}_n) \leq 0$ for all $\underset{\sim}{x}_n$ and all n , we obtain that

$$\begin{aligned} &\min\left(0,\ C_1\pi_{n+1}(\underset{\sim}{x}_{n+1}) - C_2p(1 - \pi_{n+1}(\underset{\sim}{x}_{n+1})) + M_{n+1}^{(1)}(\underset{\sim}{x}_{n+1})\right) \\ &\leq \min\left(0,\ C_1\pi_{n+1}(\underset{\sim}{x}_{n+1}) - C_2p(1 - \pi_{n+1}(\underset{\sim}{x}_{n+1}))\right) , \end{aligned} \tag{3.14}$$

for all $\underset{\sim}{x}_{n+1}$. Hence, the conditional expectations of the two sides of (3.14), given $\underset{\sim}{x}_n$, preserve the inequality. That is, $M_n^{(2)}(\underset{\sim}{x}_n) \leq M_n^{(1)}(\underset{\sim}{x}_n)$, for all $n=1,2,\ldots$ and all $\underset{\sim}{x}_n$. If we assume that $M_n^{(k)}(\underset{\sim}{x}_n) \leq M_n^{(k-1)}(\underset{\sim}{x}_n)$ for all $k=1,\ldots,i$, all $n=1,2,\ldots$, and all $\underset{\sim}{x}_n$, we obtain that

$$\begin{aligned} &\min\Big(0,\ C_1\pi_{n+1}(\underset{\sim}{x}_{n+1}) - C_2p\big(1 - \pi_{n+1}(\underset{\sim}{x}_{n+1})\big) + M_{n+1}^{(i)}(\underset{\sim}{x}_{n+1})\Big) \\ &\leq \min\Big(0,\ C_1\pi_{n+1}(\underset{\sim}{x}_{n+1}) - C_2p\big(1 - \pi_{n+1}(\underset{\sim}{x}_{n+1})\big) + M_{n+1}^{(i-1)}(\underset{\sim}{x}_{n+1})\Big) . \end{aligned} \tag{3.15}$$

The conditional expectation of the LHS of (3.15), given $\underset{\sim}{x}_n$, is $M_n^{(i+1)}(\underset{\sim}{x}_n)$ and that of the RHS is $M_n^{(i)}(\underset{\sim}{x}_n)$. Thus we have established that $M_n^{(i+1)}(\underset{\sim}{x}_n) \leq M_n^{(i)}(\underset{\sim}{x}_n)$ for all $i=1,2,\ldots$, all $n=1,2,\ldots$, and all $\underset{\sim}{x}_n$. Q.E.D.

Define the boundary values

$$b_n^{(j)}(\underset{\sim}{x}_n) = \min\left\{\frac{C_2p - M_n^{(j-1)}(\underset{\sim}{x}_n)}{C_1 + C_2p} , 1\right\}, \quad j=1,2,\ldots . \tag{3.16}$$

When there are only j optional observations it is optimal to stop sampling if $\pi_n(\underset{\sim}{x}_n) \geq b_n^{(j)}(\underset{\sim}{x}_n)$. Notice that the inequality $M_n^{(j)}(\underset{\sim}{x}_n) \leq M_n^{(j-1)}(\underset{\sim}{x}_n)$ implies that $b_n^{(j)}(\underset{\sim}{x}_n) \geq b_n^{(j-1)}(\underset{\sim}{x}_n)$ for all $j=1,2,\ldots$ with probability one. Since the sequences $b_n^{(j)}(\underset{\sim}{x}_n)$ are bounded from above by 1 , $\lim_{j\to\infty} b_n^{(j)}(\underset{\sim}{x}_n)$ exists for each n and each $\underset{\sim}{x}_n$. Denote these limits by $b_n(\underset{\sim}{x}_n)$. The stopping rule which calls for stopping if $\pi_n(\underset{\sim}{x}_n) \geq b_n(\underset{\sim}{x}_n)$ is an optimal solution of the untruncated problem. This is due to the fact that the risk is bounded below by zero (see [1]). We notice in (3.16) that for all n and all $\underset{\sim}{x}_n$, $b_n(\underset{\sim}{x}_n) \geq C_2p/(C_1 + C_2p)$.

Hence, if $\pi_n(\underset{\sim}{x}_n) < C_2p/(C_1 + C_2p)$ it is optimal to continue. The question is how to determine $b_n(\underset{\sim}{x}_n)$ when $\underset{\sim}{x}_n$ is such that $\pi_n(\underset{\sim}{x}_n) \geq C_2p/(C_1 + C_2p)$.

4. <u>The Case of Known Success Probabilities</u>

Consider the case of known success probabilities θ and ϕ . This is a special case of the general Bayesian model, in which the prior distribution of (θ,ϕ) is concentrated on a specific point (θ_0,ϕ_0) . This special case can be applied to practical control problems in which a *rectification of a process is needed whenever* $\phi \geq \phi_0$ and the interval (θ_0,ϕ_0) is a "region of indifference." Given $\underset{\sim}{x}_n$, the posterior probability of $\{\tau < n\}$ for this special model is:

$$\pi_n(\underset{\sim}{x}_n)$$

$$= 1 - \frac{(1-\pi_0)(1-p)^{n-1}}{\pi_0\left(\frac{\rho_1}{\rho_0}\right)^{T_n}\omega^n + (1-\pi_0)\,p\sum_{j=1}^{n-1}(1-p)^{j-1}\left(\frac{\rho_1}{\rho_0}\right)^{T_{n-j}^{(n)}}\omega^{n-j} + (1-\pi_0)(1-p)^{n-1}}, \tag{4.1}$$

where $\rho_1 = \phi_0/(1-\phi_0)$, $\rho_0 = \theta_0/(1-\theta_0)$, and $\omega = (1-\phi_0)/(1-\theta_0)$.

Let $\bar{\pi}_n(\underset{\sim}{x}_n) = 1 - \pi_n(\underset{\sim}{x}_n)$. We can easily establish the recursive relationship:

$$\bar{\pi}_{n+1}(\underset{\sim}{x}_{n+1}) = \frac{\bar{\pi}_n(\underset{\sim}{x}_n)(1-p)}{\left(1-\bar{\pi}_n(\underset{\sim}{x}_n)\right)\left(\frac{\rho_1}{\rho_0}\right)^{x_{n+1}}\omega + \bar{\pi}_n(\underset{\sim}{x}_n)p\left(\frac{\rho_1}{\rho_0}\right)^{x_{n+1}}\omega + \bar{\pi}_n(\underset{\sim}{x}_n)(1-p)} . \tag{4.2}$$

This recursive relationship shows that when θ and ϕ are known, the $\{\pi_n(\underset{\sim}{x}_n);\ n \geq 1\}$ process is Markovian. In order to determine the value of $\pi_{n+1}(\underset{\sim}{x}_{n+1})$ it is sufficient to know the value of $\pi_n(\underset{\sim}{x}_n)$ and the

value of x_{n+1} . Let $\pi_n(\underset{\sim}{x}_n) = \pi$ and $X_{n+1} = Y$, and let

$$\psi(\pi,Y) = \frac{Z^Y\omega\big(\pi + (1-\pi)p\big)}{Z^Y\omega\big(\pi + (1-\pi)p\big) + (1-\pi)(1-p)} , \qquad (4.3)$$

where $Z = \rho_1/\rho_0$ and $\psi(\pi,Y)$ is $\pi_{n+1}(\underset{\sim}{x}_n,Y)$ as a function of π and Y . Thus, the sequence $\{\pi_n(\underset{\sim}{x}_n);\ n \geq 1\}$ is a stationary Markov sequence in the sense that, given $\pi_n(\underset{\sim}{x}_n) = \pi$ the distribution $\pi_{n+k}(\underset{\sim}{x}_{n+k})$ for all $k \geq 1$ is independent of n . Furthermore, one can readily prove that $\{\pi_n(\underset{\sim}{x}_n);\ n \geq 1\}$ is a submartingale with respect to the Bayes predictive distributions of $\underset{\sim}{x}_n$ $(n \geq 1)$ [see the denominator on the RHS of (4.1)]. Thus, with respect to these Bayes predictive distributions, $\pi_n(\underset{\sim}{x}_n) \to 1$ a.s. (see Sirjaev [2, p. 153]). We provide in the following lemma a proof for our specific problems which establishes the convergence of $\pi_n(\underset{\sim}{x}_n)$ to one whenever a shift occurs at a fixed finite time point $\tau = k$. The convergence established in Lemma 2 is with respect to a sequence of distributions with fixed parameters, while Sirjaev's result is the convergence a.s. with respect to the prior mixtures of such distributions.

<u>Lemma 2</u>: When θ_0 and ϕ_0 are known, and if $\tau = k$ for some $k < \infty$, then $\pi_n(\underset{\sim}{x}_n) \to 1$ a.s. as $n \to \infty$.

<u>Proof</u>: Let

$$S_n = \frac{\pi_n(\underset{\sim}{x}_n)}{1 - \pi_n(\underset{\sim}{x}_n)} .$$

It is sufficient to show that $S_n \to \infty$ a.s. $[\phi_0]$. According to (4.1),

$$S_n = \frac{\pi_0}{1 - \pi_0}\left(\frac{\omega}{1-p}\right)^n (1-p)Z^{T_n} + p\sum_{j=1}^{n-1}\left(\frac{\omega}{1-p}\right)^{n-j} Z^{T_{n-j}^{(n)}} . \qquad (4.4)$$

Obviously,

$$S_{n+1} = (S_n+p)Z^{X_{n+1}}\,\frac{\omega}{1-p} \geq S_n Z^{X_{n+1}}\,\frac{\omega}{1-p} ;$$

hence

$$S_{n+k} \geq S_k \, Z^{T_n^{(n+k)}} \left(\frac{\omega}{1-p}\right)^n$$

$$= \frac{S_k}{(1-p)^n} \frac{\phi_0^{T_n^{(n+k)}} (1-\phi_0)^{n-T_n^{(n+k)}}}{\theta_0^{T_n^{(n+k)}} (1-\theta_0)^{n-T_n^{(n+k)}}} \; . \tag{4.5}$$

The function $\omega^{T_n}(1-\omega)^{n-T_n}$ is maximized by $\hat{\omega} = T_n/n$. Hence

$$\frac{\left(\frac{T_n^{(n+k)}}{n}\right)^{T_n^{(n+k)}} \left(1 - \frac{T_n^{(n+k)}}{n}\right)^{n-T_n^{(n+k)}}}{\theta_0^{T_n^{(n+k)}} (1-\theta_0)^{n-T_n^{(n+k)}}} \geq 1 \; , \tag{4.6}$$

for all θ in $(0,1)$. Finally, since $T_n^{(n+k)}/n \to \phi_0$ a.s., one obtains

$$\lim_{n\to\infty} S_{n+k} \geq \lim_{n\to\infty} \frac{S_k}{(1-p)^n} \lim_{n\to\infty} \frac{\phi_0^{T_n^{(n+k)}} (1-\phi_0)^{n-T_n^{(n+k)}}}{\theta_0^{T_n^{(n+k)}} (1-\theta_0)^{n-T_n^{(n+k)}}} \tag{4.7}$$

$$= \lim_{n\to\infty} \frac{S_k}{(1-p)^n} \lim_{n\to\infty} \frac{\left(\frac{T_n^{(n+k)}}{n}\right)^{T_n^{(n+k)}} \left(1 - \frac{T_n^{(n+k)}}{n}\right)^{n-T_n^{(n+k)}}}{\theta_0^{T_n^{(n+k)}} (1-\theta_0)^{n-T_n^{(n+k)}}}$$

$$= \infty \; , \quad \text{a.s. } [\phi_0] \; .$$

Q.E.D.

The boundary function for the optimal stopping rule depends on the value of π and does not depend on n . Let $B(\pi)$, $0 \leq \pi \leq 1$, denote the boundary function. The stopping rule requires that sampling be terminated as soon as $\pi_n(\underset{\sim}{x}_n) \geq B(\pi_n(\underset{\sim}{x}_n))$. As proven earlier, $B(\pi) \geq \pi^*$,

where $\pi^* = C_2p/(C_1 + C_2p)$. Let $M^{(i)}(\pi)$ denote the function $M_n^{(i)}(\underset{\sim}{x}_n)$, $i=1,2,\ldots$ and let $M(\pi)$ denote the function $M_n(\underset{\sim}{x}_n)$ when $\pi_n(\underset{\sim}{x}_n) = \pi$. These functions do not depend on n . The boundary $B(\pi)$ is defined as

$$B(\pi) = \min\left(1 \ , \ \frac{C_2p - M(\pi)}{C_1 + C_2p}\right), \qquad 0 \leq \pi \leq 1 \ . \tag{4.8}$$

Similarly, for each $i=1,2,\ldots$ define

$$B^{(i)}(\pi) = \min\left(1 \ , \ \frac{C_2p - M^{(i)}(\pi)}{C_2p + C_1}\right) . \tag{4.9}$$

The sequence $\{B^{(i)}(\pi);\ i=1,2,\ldots\}$ converges monotonically to $B(\pi)$ for each π . Consider the functions $M^{(1)}(\pi)$ and $B^{(1)}(\pi)$. According to (3.7),

$$M^{(1)}(\pi) = E_\pi\left\{\min\left(0 \ , \ C_1\psi(\pi,X) - C_2p\left(1 - \psi(\pi,X)\right)\right)\right\} , \tag{4.10}$$

where the distribution of X has the probability function

$$p(x;\ \pi) \quad = \quad [\pi + (1-\pi)p]\phi_0^x \ (1-\phi_0)^{1-x} + (1-\pi)^{(1-p)}\theta_0^x \ (1-\theta_0)^{1-x} \ . \tag{4.11}$$

We can easily verify that for each value of X , $\psi(\pi,X)$ is a strictly increasing function of π and that $\psi(\pi,1) > \pi$ for every π in $(0,1)$. Thus, if $\pi \geq \pi^*$, then $\psi(\pi,1) \geq \pi^*$, and therefore $C_1\psi(\pi,1) - C_2p(1 - \psi(\pi,1)) \geq 0$. Hence, for $\pi \geq \pi^*$,

$$M^{(1)}(\pi) = P_\pi\{X=0\}\left(C_1\psi(\pi,0) - C_2p\left(1 - \psi(\pi,0)\right)\right)^- , \tag{4.12}$$

where $a^- = \min(0,a)$; and $P_\pi\{X=0\} = \pi(1-\phi_0) + (1-\pi)(1-\theta_0)$. Substituting the formula for $\psi(\pi,0)$ into (4.6) yields

$$M^{(1)}(\pi) = \left[\pi(1-\phi_0) + (1-\pi)(1-\theta_0)\right] \frac{\left[C_1\omega\left(\pi+p(1-\pi)\right) - C_2p(1-\pi)(1-p)\right]^-}{\omega\left(\pi+p(1-\pi)\right) + (1-\pi)(1-p)} \ . \tag{4.13}$$

Notice that $M^{(1)}(\pi)$ is a continuous function of π and that $M^{(1)}(\pi) \to 0$ as $\pi \to 1$. Correspondingly, $B^{(1)}(\pi)$ is a continuous function of π and $B^{(1)}(\pi) \to \pi^*$ as $\pi \to 1$.

According to the recursive equation (3.12),

$$M^{(2)}(\pi) = E_\pi\left\{\left(C_1\psi(\pi,X) - C_2p\left(1-\psi(\pi,X)\right) + M^{(1)}\left(\psi(\pi,X)\right)\right)^-\right\} . \quad (4.14)$$

Thus, $M^{(2)}(\pi)$ is a continuous function of π. In a close neighborhood of 1, $M^{(1)}\left(\psi(\pi,X)\right) = 0$ for $x=0,1$ and therefore $M^{(2)}(\pi) = M^{(1)}(\pi)$. By induction of i we show that $M^{(i)}(\pi)$ is a continuous function of π for all $i=1,2,\ldots$ and that $M^{(i)}(\pi) \to 0$ as $\pi \to 1$. Correspondingly, all the boundary functions $B^{(i)}(\pi)$ are continuous and converge to π^* as $\pi \to 1$. Thus we can show that $B(\pi)$ is continuous and converges to π^* as $\pi \to 1$. Therefore, there exists a value π_1 such that $\pi_1 = B(\pi_1)$ and for the first n at which $\pi_n(\underset{\sim}{x}_n) \geq \pi_1$ it is optimal to stop. Sirjaev [2, pp. 149-155] proved this result in a more general context; however, he has not determined the value of π_1. We have shown that in our framework, $\pi_1 \geq \pi^*$.

5. Unknown Success Probabilities with Uniform Prior

In this section we investigate the nature of the decision process when the prior distribution of the unknown (θ,ϕ) is uniform over the simplex $0 < \theta \leq \phi < 1$, i.e., the prior p.d.f. is

$$h(\theta,\phi) = 2I\{0 < \theta \leq \phi < 1\} . \quad (5.1)$$

The posterior p.d.f. of (θ,ϕ) is

$$g(\theta,\phi \mid \underset{\sim}{x}_n) = I\{0 < \theta \leq \phi < 1\} \frac{\sum_{j=0}^{\infty} \psi(j)L(j,\theta,\phi;\ \underset{\sim}{x}_n)}{D_n(\underset{\sim}{x}_n)} . \quad (5.2)$$

Here

$$\sum_{j=0}^{\infty} \psi(j)L(j,\theta,\phi;\, \underset{\sim}{x}_n) = \pi\, \phi^{T_n} (1-\phi)^{n-T_n}$$

$$+ (1-\pi)p \sum_{j=1}^{n-1} (1-p)^{j-1}\, \theta^{T_j} (1-\theta)^{j-T_j}\, \phi^{T_{n-j}^{(n)}} (1-\phi)^{n-j-T_{n-j}^{(n)}}$$

$$+ (1-\pi)(1-p)^{n-1}\, \theta^{T_n} (1-\theta)^{n-T_n} , \qquad (5.3)$$

and the function $D_n(\underset{\sim}{x}_n)$ is obtained by integrating (5.3) over the range $0 \le \theta \le \phi \le 1$. Accordingly, we obtain after some algebraic manipulations

$$D_n(\underset{\sim}{x}_n) = \pi\, B(T_n+2 \ , \ n-T_n+1)$$

$$+ (1-\pi)p \sum_{j=1}^{n-1} (1-p)^{j-1} \frac{B(T_{n-j}^{(n)}+1 \ , \ n-j-T_{n-j}^{(n)}+1)}{n-j+2} \sum_{i=0}^{T_{n-j}^{(n)}} \frac{B(T_n-i+1 \ , \ n-T_n+i+2)}{B(T_{n-j}^{(n)}-i+1 \ , \ n-j-T_{n-j}^{(n)}+i+2)}$$

$$+ (1-\pi)(1-p)^{n-1}\, B(T_n+1 \ , \ n-T_n+2) , \qquad (5.4)$$

where $B(\nu_1,\nu_2) = \Gamma(\nu_1)\Gamma(\nu_2)/\Gamma(\nu_1+\nu_2)$ is the beta function. The posterior probability $\pi_n(\underset{\sim}{x}_n)$ can be determined by the formula

$$\pi_n(\underset{\sim}{x}_n) = 1 - (1-\pi)(1-p)^{n-1}\, B(T_n+1 \ , \ n-T_n+2)/D_n(\underset{\sim}{x}_n) . \qquad (5.5)$$

If we denote by Y the result of the (n+1)st trial, and if $\bar{\pi}_n(\underset{\sim}{x}_n) = 1 - \pi_n(\underset{\sim}{x}_n)$, then we obtain from (5.5) the expression

$$\bar{\pi}_{n+1}(\underset{\sim}{x}_n,Y) = \frac{(1-p)\, \bar{\pi}_n(\underset{\sim}{x}_n)\, D_n(\underset{\sim}{x}_n)\, B(T_n+1+Y \ , \ n+3-T_n-Y)}{D_{n+1}(\underset{\sim}{x}_n,Y)\, B(T_n+1 \ , \ n-T_n+2)} . \qquad (5.6)$$

More specifically,

$$\bar{\pi}_{n+1}(\underset{\sim}{x}_n,1) = (1-p)\bar{\pi}_n(\underset{\sim}{x}_n) \frac{T_n+1}{n+3} \cdot \frac{D_n(\underset{\sim}{x}_n)}{D_{n+1}(\underset{\sim}{x}_n,1)} \qquad (5.7)$$

and

$$\bar{\pi}_{n+1}(\underset{\sim}{x}_n,0) = (1-p)\bar{\pi}_n(\underset{\sim}{x}_n)\,\frac{n+2-T_n}{n+3}\cdot\frac{D_n(\underset{\sim}{x}_n)}{D_{n+1}(\underset{\sim}{x}_n,0)}\,. \tag{5.8}$$

From the basic definitions we can establish that

$$P[x_{n+1}=1 \mid \underset{\sim}{x}_n] = \frac{D_{n+1}(\underset{\sim}{x}_n,1)}{D_n(\underset{\sim}{x}_n)}\,. \tag{5.9}$$

According to (3.3), the process $\{\pi_n(\underset{\sim}{x}_n);\ n \geq 1\}$ constitutes a submartingale with respect to the Bayes predictive distributions of $\underset{\sim}{x}_n$. Thus, $\lim_{n\to\infty} \pi_n(\underset{\sim}{x}_n)$ exists with probability one, and as in Sirjaev [2, p. 155] we can show that $\lim_{n\to\infty} \pi_n(\underset{\sim}{x}_n) = 1$ a.s., with respect to the Bayes predictive distributions. In the following lemma we prove this convergence for cases of fixed (θ_0,ϕ_0) and $\tau = k$ (finite).

<u>Lemma 3</u>: If $\tau = k$ for some $k < \infty$ then $\pi_n(\underset{\sim}{x}_n) \to 1$ a.s. as $n \to \infty$.

<u>Proof</u>: We write

$$\pi_n(\underset{\sim}{x}_n) = \int_{\theta=0}^{1}\int_{\phi=\theta}^{1} P[\tau<n-1 \mid \underset{\sim}{x}_n,\theta,\phi]\ g(\theta,\phi \mid \underset{\sim}{x}_n)d\phi d\theta\,. \tag{5.10}$$

We see in (4.7) that as $n \to \infty$ then $P[\tau<n-1 \mid \underset{\sim}{x}_n,\theta,\phi] \to 1$ a.s., uniformly in (θ,ϕ). Accordingly, there exists $N(\delta)$ such that for all $n \geq N(\delta)$, $P[\tau<n-1 \mid \underset{\sim}{x}_n,\theta,\phi] \geq 1-\delta$. Thus, from (5.10),

$$\pi_n(\underset{\sim}{x}_n) \geq (1-\delta)\int_0^1\int_\theta^1 g(\theta,\phi \mid \underset{\sim}{x}_n)d\phi d\theta = 1-\delta\,. \tag{5.11}$$

Q.E.D.

Note that the above proof does not depend on the assumption of the uniform prior distribution of (θ,ϕ).

6. Some Numerical Examples

In this section we provide several numerical illustrations of the stopping rule:

$$N = \text{least } n \geq 1 \text{ such that } \pi_n(\underset{\sim}{x}_n) \geq b_n^{(1)}(\underset{\sim}{x}_n) , \tag{6.1}$$

for the case where θ and ϕ have a uniform prior on the simplex $0 \leq \theta \leq \phi \leq 1$, and the shift parameter τ has a geometric prior distribution. Each illustration is based on an independent simulation of X values, in which X_n is a Bernoulli random variable with a specified parameter θ if $n \leq \tau$ and with parameter ϕ otherwise. The simulation in a given run is continued until the decision rule calls for stopping. One hundred independent replicas were run in each case, and the empirical frequency distribution of the stopping locations was recorded. In Table 1 we present these frequency distributions for cases in which $\tau = 10$, $\pi = 0.01$, $p = 0.01$, $\theta = 0.3$, and $\lambda = C_1/C_2 = 0.06$. We varied the parameters ϕ over the range .5 to 1. in order to illustrate the effect of ϕ on the speed of detection. The above parameters π and p were chosen small in order to lessen their effect on the stopping times. The value of λ was chosen sufficiently small to reduce early stopping.

As indicated in Table 1, the distribution of stopping locations after the shift has occurred tends to concentrate near the point of shift as ϕ increases. This is expected, since large values of ϕ frequently yield the value $x = 1$. On the whole, it seems that the stopping rule (6.1) is sensitive and its performance can be controlled by varying the parameters π , p , and λ . The number of replicas on which Table 1 is based is too small for definitive comparisons of the stopping time distributions. To establish these distributions more accurately, either extensive simulations or a different numerical approach is needed.

TABLE 1

EMPIRICAL FREQUENCY DISTRIBUTIONS
OF STOPPING RULE (6.1)

n \ ϕ	0.5[a]	0.6	0.7	0.8	0.9	1.0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	3	2	0	3	4	5
6	0	0	0	0	1	1
7	5	4	5	7	5	6
8	6	5	3	5	3	4
9	3	2	3	2	2	1
10	4	8	6	7	7	7
11	11	13	12	7	10	10
12	11	8	15	23	31	35
13	7	20	15	15	26	28
14	6	7	11	15	6	3
15	6	5	10	7	3	0
16	5	7	8	5	2	0
17	11	6	3	2	0	0
18	3	1	3	1	0	0
19	4	3	4	1	0	0
20	4	4	2	0	0	0
21	2	1	0	0	0	0
22	1	0	0	0	0	0
23	3	2	0	0	0	0
24	0	1	0	0	0	0
25	1	0	0	0	0	0
26	1	1	0	0	0	0
27	1	0	0	0	0	0
28	0	0	0	0	0	0
29	1	0	0	0	0	0
30	0	0	0	0	0	0

[a]In one case here the decision rule did not call for stopping even after 30 observations.

REFERENCES

[1] CHOW, Y. S., H. P. ROBBINS and D. SIEGMUND (1971). Great Expectations: The Theory of Optimal Stopping. Houghton Mifflin Company, Boston.

[2] SIRJAEV, A. N. (1973). Statistical Sequential Analysis - Optimal Stopping Rules. Translations of mathematical monographs, **38**, American Mathematical Society, Providence, Rhode Island. (Lisa and Judah Rosenblatt, translators).

THE GEORGE WASHINGTON UNIVERSITY
Program in Logistics
Distribution List for Technical Papers

The George Washington University
- Office of Sponsored Research
- Library
- Vice President H. F. Bright
- Dean Harold Liebowitz
- Mr. J. Frank Doubleday

ONR
- Chief of Naval Research
  - (Codes 200, 430D, 1021P)
  - Resident Representative

OPNAV
- OP-40
- DCNO, Logistics
- Navy Dept Library
- OP-911
- OP-964

Naval Aviation Integrated Log Support

NAVCOSSACT

Naval Cmd Sys Sup Activity Tech Library

Naval Electronics Lab Library

Naval Facilities Eng Cmd Tech Library

Naval Ordnance Station
- Louisville, Ky.
- Indian Head, Md.

Naval Ordnance Sys Cmd Library

Naval Research Branch Office
- Boston
- Chicago
- New York
- Pasadena
- San Francisco

Naval Research Lab
- Tech Info Div
- Library, Code 2029 (ONRL)

Naval Ship Engng Center
- Philadelphia, Pa.
- Hyattsville, Md.

Naval Ship Res & Dev Center

Naval Sea Systems Command
- Tech Library
- Code 073

Naval Supply Systems Command
- Library
- Capt W. T. Nash

Naval War College Library
- Newport

BUPERS Tech Library

FMSO

Integrated Sea Lift Study

USN Ammo Depot Earle

USN Postgrad School Monterey
- Library
- Dr. Jack R. Borsting
- Prof C. R. Jones

US Marine Corps
- Commandant
- Deputy Chief of Staff, R&D

Marine Corps School Quantico
- Landing Force Dev Ctr
- Logistics Officer

Armed Forces Industrial College

Armed Forces Staff College

Army War College Library
- Carlisle Barracks

Army Cmd & Gen Staff College

US Army HQ
- LTC George L. Slyman
- Army Trans Mat Command

Army Logistics Mgmt Center
- Fort Lee

Commanding Officer, USALDSRA
- New Cumberland Army Depot

US Army Inventory Res Ofc
- Philadelphia

HQ, US Air Force
- AFADS-3

Griffiss Air Force Base
- Reliability Analysis Center

Maxwell Air Force Base Library

Wright-Patterson Air Force Base
- HQ, AF Log Command
- Research Sch Log

Defense Documentation Center

National Academy of Science
- Maritime Transportation Res Board Library

National Bureau of Standards
- Dr E. W. Cannon
- Dr Joan Rosenblatt

National Science Foundation

National Security Agency

WSEG

British Navy Staff

Logistics, OR Analysis Establishment
- National Defense Hdqtrs, Ottawa

American Power Jet Co
- George Chernowitz

ARCON Corp

General Dynamics, Pomona

General Research Corp
- Dr Hugh Cole
- Library

Planning Research Corp
- Los Angeles

Rand Corporation
- Library

Carnegie-Mellon University
- Dean H. A. Simon
- Prof G. Thompson

Case Western Reserve University
- Prof B. V. Dean
- Prof John R. Isbell
- Prof M. Mesarovic
- Prof S. Zacks

Cornell University
- Prof R. E. Bechhofer
- Prof R. W. Conway
- Prof J. Kiefer
- Prof Andrew Schultz, Jr.

Cowles Foundation for Research
- Library
- Prof Herbert Scarf
- Prof Martin Shubik

Florida State University
- Prof R. A. Bradley

Harvard University
- Prof K. J. Arrow
- Prof W. G. Cochran
- Prof Arthur Schleifer, Jr.

New York University
- Prof O. Morgenstern

Princeton University
- Prof A. W. Tucker
- Prof J. W. Tukey
- Prof Geoffrey S. Watson

Purdue University
Prof S. S. Gupta
Prof H. Rubin
Prof Andrew Whinston

Stanford
Prof T. W. Anderson
Prof G. B. Dantzig
Prof F. S. Hillier
Prof D. L. Iglehart
Prof Samuel Karlin
Prof G. J. Lieberman
Prof Herbert Solomon
Prof A. F. Veinott, Jr.

University of California, Berkeley
Prof R. E. Barlow
Prof D. Gale
Prof Rosedith Sitgreaves
Prof L. M. Tichvinsky

University of California, Los Angeles
Prof J. R. Jackson
Prof Jacob Marschak
Prof R. R. O'Neill
Numerical Analysis Res Librarian

University of North Carolina
Prof W. L. Smith
Prof M. R. Leadbetter

University of Pennsylvania
Prof Russell Ackoff
Prof Thomas L. Saaty

University of Texas
Prof A. Charnes

Yale University
Prof F. J. Anscombe
Prof I. R. Savage
Prof M. J. Sobel
Dept of Admin Sciences

Prof Z. W. Birnbaum
University of Washington

Prof B. H. Bissinger
The Pennsylvania State University

Prof Seth Bonder
University of Michigan

Prof G. E. P. Box
University of Wisconsin

Dr. Jerome Bracken
Institute for Defense Analyses

Prof H. Chernoff
MIT

Prof Arthur Cohen
Rutgers – The State University

Mr Wallace M. Cohen
US General Accounting Office

Prof C. Derman
Columbia University

Prof Paul S. Dwyer
Mackinaw City, Michigan

Prof Saul I. Gass
University of Maryland

Dr Donald P. Gaver
Carmel, California

Dr Murray A. Geisler
Logistics Mgmt Institute

Prof J. F. Hannan
Michigan State University

Prof H. O. Hartley
Texas A & M Foundation

Mr Gerald F. Hein
NASA, Lewis Research Center

Prof W. M. Hirsch
Courant Institute

Dr Alan J. Hoffman
IBM, Yorktown Heights

Dr Rudolf Husser
University of Bern, Switzerland

Prof J. H. K. Kao
Polytech Institute of New York

Prof W. Kruskal
University of Chicago

Prof C. E. Lemke
Rensselaer Polytech Institute

Prof Loynes
University of Sheffield, England

Prof Steven Nahmias
University of Pittsburgh

Prof D. B. Owen
Southern Methodist University

Prof E. Parzen
State University New York, Buffalo

Prof H. O. Posten
University of Connecticut

Prof R. Remage, Jr.
University of Delaware

Dr Fred Rigby
Texas Tech College

Mr David Rosenblatt
Washington, D. C.

Prof M. Rosenblatt
University of California, San Diego

Prof Alan J. Rowe
University of Southern California

Prof A. H. Rubenstein
Northwestern University

Dr M. E. Salveson
West Los Angeles

Prof Edward A. Silver
University of Waterloo, Canada

Prof R. M. Thrall
Rice University

Dr S. Vajda
University of Sussex, England

Prof T. M. Whitin
Wesleyan University

Prof Jacob Wolfowitz
University of Illinois

Mr Marshall K. Wood
National Planning Association

Prof Max A. Woodbury
Duke University